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MATHEMATICAL MODELS IN THE DESIGN OF FLAT OPTICS ELEMENTS

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Abstract—Mathematical problems occurring in the solution of inverse problems of flat-optics design are considered. Effective algorithms are proposed for solving such problems as the formation of directivity diagrams, design of a given flat image, or focusing of electromagnetic radiation.

INTRODUCTION

We consider the following inverse problems in optical element design:

Problem 1. Generating a specified image in a given plane.
Problem 2. Generating specified directivity diagrams.
Problem 3. Focusing electromagnetic radiation onto a given curve.
Problem 4. Construction of flat optics elements on a diffraction lattice (couplers, beam splitter).

The mathematical problems which arise in solving the above design goals may all be reduced to the solution of the operator equation

$$A\varphi = F \tag{1}$$

where F describes the image, φ the optical element, and the operator A establishes the correspondence between φ and F in a given mathematical model (φ and F are elements in a space of functionals, taken as normalized for the sake of concreteness). The following mathematical problems arise:

1. Choice of mathematical model

On the one hand, the model must be simple enough to be tractable, on the other, it should reflect to the required accuracy the actual processes occurring in the optical system.

2. The solubility of the synthesis

The synthesis problem represented by Eq. (1) is soluble if for any F there exists an element $\varphi \in Q$, such that

$$\inf_{\varphi \in Q} \|A\varphi - F\| = 0.$$

Here Q is the manifold of possible realizations. There must exist an element φ_ε which solves the synthesis problem to a specified accuracy ε, that is to say $\|A\varphi_\varepsilon - F\| \leqslant \varepsilon$. In that case one speaks of an ε-soluble synthesis problem.

3. Uniqueness problem

Let us remark right away that in the majority of cases the synthesis problem does not have a unique solution. Nevertheless, not as in diagnostics, the nonunique nature of the solution plays a positive role in that it can be used to optimize additional element characteristics.

4. Construction of effective algorithms to solve the inverse problem of optical element synthesis

We remark that the search is for a stable algorithm capable of providing solutions with specified accuracy.

5. The creation of suitable software/hardware for computer-aided design of optical elements

MATHEMATICAL MODELS IN OPTICAL ELEMENT DESIGN

The Fresnel approximation

One of the most widespread models in optical element design is the Fresnel approximation, wherein the radiation propagation in vacuum is given by the wave function

$$U(x, y, z, t) = u(x, y, z)e^{-i\omega t}.$$

Here $u(x, y, z)$ is a complex wave field satisfying the Helmholtz equation

$$\Delta u + k^2 u = 0.$$

We assume that the optical element is in the $z = 0$ plane, and that $u(\xi, \eta, 0-0)$ is the wave function of the radiation incident on the $z = 0$ plane. Let $u(\xi, \eta, 0+0)$ be the wave function emerging from the flat optical element. Without loss of generality we may take the flat optical element to be perpendicular to the incident wave (Fig. 1) (e.g. this could be a plane electromagnetic wave propagating along the z-axis). In the flat optics approximation the following expression relates the complex wave function $u(\xi, \eta, 0+0)$ to the field of $u(\xi, \eta, 0-0)$:

$$u(\xi, \eta, 0+0) = u(\xi, \eta, 0-0)C(\xi, \eta). \tag{2}$$

In (2) $C(\xi, \eta)$ is the complex transmission function of the element. If $|C(\xi, \eta)| \equiv 1$ in region G, we have a phase element. If Im $C(\xi, \eta) = 0$ in G, one speaks of an amplitude element. It follows from (1) that knowledge of $u(\xi, \eta, 0-0)$ and $u(\xi, \eta, 0+0)$ enables one to determine, subject to (2), the transmissivity $C(\xi, \eta)$ and to produce the optical element, thereby solving the problem of synthesis [1]. The problem is thus reduced to calculating the wave field $u(\xi, \eta, 0+0)$. Let us write $u(\xi, \eta, 0+0) = A(\xi, \eta)e^{ik\varphi(\xi,\eta)}$, where $A(\xi, \eta)$ is real. The Fresnel approximation for the modulus of the complex wave field $u(x, y, f)$ in the plane $z = f$ is given by

$$|u(x, y, f)| = \left| \frac{k}{2\pi} \int_G \int e^{ik(x\xi/f + y\eta/f)} e^{ik((\xi^2+\eta^2)/2f)} \cdot A(\xi, \eta) e^{ik\varphi(\xi,\eta)} \, d\xi \, d\eta \right| = B(x, y). \tag{3}$$

In the problem of synthesis as formulated in (1) the function $B(x, y)$ is given and $A(\xi, \eta)$, $\varphi(\xi, \eta)$ are to be found from (3). Denoting the pair of functions A, φ by Φ, Eq. (3) may be rewritten in operator form

$$T\Phi = B. \tag{4}$$

For a phase element, $A(\xi, \eta)$ is known and only $\varphi(\xi, \eta)$ has to be determined.

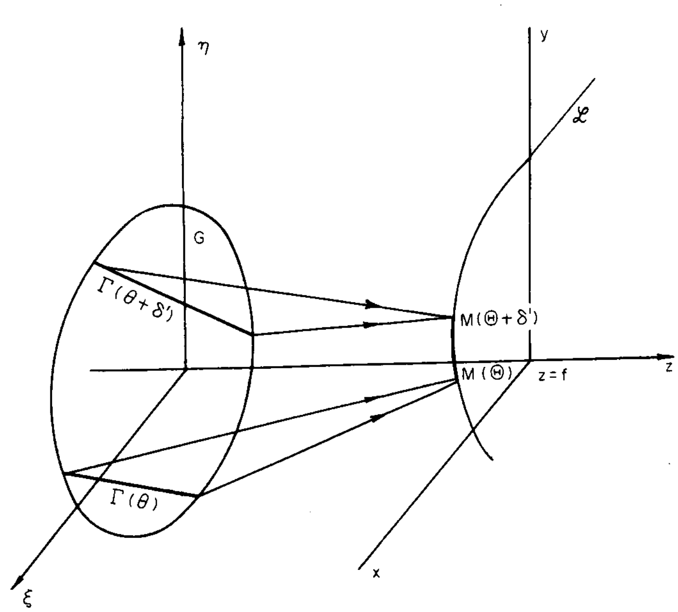

Fig. 1. Geometric arrangement of flat optical element and the region of image formation (problems 1 and 3).

Problem 2—the formation of directivity diagrams—actually reduces to *problem 1*. Let us consider Eq. (3) in the approximation that $f \to \infty$, while $x/f = \alpha$, $y/f = \beta$, where α, β are direction cosines of the normal vector. Then

$$\lim_{f\to\infty} |fu(x, y, f)| = \frac{k}{2\pi}\left|\int_G\int A(\xi, \eta)\mathrm{e}^{\mathrm{i}k\varphi(\xi,\eta)}\cdot\mathrm{e}^{\mathrm{i}k(\xi\alpha+\eta\beta)}\,\mathrm{d}\xi\,\mathrm{d}\eta\right| = R(\alpha, \beta). \tag{5}$$

The function $R(\alpha, \beta)$ depends only on α and β and is the amplitude of the directivity diagram of the optical element. Problems (1) and (2) are clearly related to one another, in the sense that if $A(\xi, \eta)$ and $\varphi(\xi, \eta)$ solve problem (1), then the functions $A(\xi, \eta)$ and

$$\tilde{\varphi}(\xi, \eta) = \varphi(\xi, \eta) + \frac{\xi^2 + \eta^2}{2f} \tag{6}$$

solve the problem of forming the directivity diagram with $R(\alpha, \beta) = \lim_{f\to\infty} |fu(\alpha f, \beta f, f)|$. Clearly $\tilde{\varphi}$ differs from φ by a factor containing the focal length of the optical lens system.

UNIQUENESS THEOREM ON SYNTHESIS PROBLEMS 1 AND 2

Let $B(x, y)$ be a square integrable function in G. Define $\Delta(A, \varphi)$ by

$$\Delta(A, \varphi) = \int\int \{B(x, y) - |u(x, y, f)|\}^2\,\mathrm{d}x\,\mathrm{d}y, \tag{7}$$

where the integration is carried out over the whole plane $z = f$. Then the following theorem holds:

Theorem 1

Square integrable functions $B(x, y)$ exist, satisfying

$$\inf_{\varphi, A} \Delta(A, \varphi) = \Delta_0 > 0.$$

Remark 1. This means that the synthesis problem of optical elements forming an image in the $z = f$ plane is far from being always unique even for amplitude-phase elements, let alone for phase optical elements. One may form a lower bound on Δ_0, which will characterize the level of approximation that in principle cannot be bettered in forming the given image F. The same results hold true for problem 2.

Remark 2. An example of such a function is any discontinuous function, such as

$$B(x, y) = \begin{cases} 1, & x^2 + y^2 \leqslant R^2 \\ 0, & x^2 + y^2 > R^2. \end{cases}$$

It can be shown that for a plane monochromatic wave ($u(\xi, \eta, z) = \mathrm{e}^{\mathrm{i}kz}$) incident on an optical element of focal length $f \sim 50$ cm, circular aperture ~ 1 cm, image diameter ~ 1 cm and $\lambda = 10.6$ μm, the magnitude of Δ is of order 1–2%.

ALGORITHMS FOR SOLVING PROBLEMS 1 AND 2 WITHIN THE FRESNEL APPROXIMATION

All existing algorithms for solving problems 1 and 2 within the Fresnel approximation reduce, one way or the other, to minimizing the difference functional

$$\Delta(A, \varphi) = \int\int \{B(x, y) - |u(x, y, f)|\}^2\,\mathrm{d}x\,\mathrm{d}y.$$

Normally one employs various iteration algorithms [2]. We point out right away that the operator T in (4) depends nonlinearly on A and φ, so that the minimization of the functional $\Delta(A, \varphi)$ turns out to be a nonconvex programming problem. The construction of an iteration algorithm ensuring convergence to the minimum of the functional $\Delta(A, \varphi)$ is an extremely complex task.

In current optics literature the iteration algorithms developed in [3] have gained wide acceptance. Introducing $v(\xi, \eta) = A(\xi, \eta)e^{ik\varphi(\xi,\eta)}$ we may write

$$u(x, y, f) = \gamma_0 e^{ik((x^2+y^2)/2f)}\mathscr{F}(v(\xi, \eta)e^{ik((\xi^2+\eta^2)/2f)}. \tag{8}$$

Here $\mathscr{F}$ denotes the two-dimensional Fourier transformation of its argument and γ_0 is a constant. In building the iteration sequence to minimize the functional $\Delta(A, \varphi)$ one may therefore employ well-established Fourier techniques (fast Fourier transforms, special processors etc.). Without loss of generality we may state that solving problems 1 and 2 for the phase elements is equivalent to the following mathematical problem. Let two functions $u(\xi, \eta)$, $v(\xi, \eta)$ be related via $u = \mathscr{F}v$, whereby $|u(\xi, \eta)| = B(\xi, \eta)$ and $|v(\xi, \eta)| = A(\xi, \eta)$ are known. The problem is then to find the phase of one of the functions, say $u(\xi, \eta)$. The algorithm presented here is essentially one of simple iteration to minimize $\Delta(A, \varphi)$, and consists of the following. Suppose the kth stage of the algorithm gives $u_k(\xi, \eta)$. Then u'_{k+1} will be calculated by the following rule: the phase of the function u_k is assigned to the amplitude $B(\xi, \eta)$, in other words $u'_{k+1}(\xi, \eta) = (u_k/|u_k|)B(\xi, \eta)$. Now $v'_{k+1} = \mathscr{F}u'_{k+1}$. The function v_{k+1} is found by the same method: $v_{k+1} = (v'_{k+1}/|v'_{k+1}|)A(x, y)$. Finally, $u_{k+1} = \mathscr{F}v_{k+1}$, and so on.

Such an algorithm has been shown to be relaxational, i.e. the difference $\|A - |v_k|\|^2_{L_2}$ is a monotonic function of the iteration k. We note that this is by no means equivalent to the statement that the sequence u_k produced by this algorithm is convergent.

To minimize the functional $\Delta(A, \varphi)$, traditional algorithms may be used, such as steepest descent, or others. Calculating the gradient of $\Delta(A, \varphi)$ presents no difficulties.

We note that in the visible range geometrical optics is normally an excellent approximation [4], and the improvement achieved by using the Fresnel approximation instead is minimal. Thus for a characteristic geometry, taking the diameter of $G \sim 1$ cm, the image diameter ~ 1 cm, $\lambda = 10.6$ μm, $f \sim 50$ cm, the difference functional $\Delta(A, \varphi)$ obtained in the first approximation represents an improvement of only a factor of 1.2–1.5 over the usual geometrical optics result.

MATHEMATICAL MODEL IN THE INVERSE PROBLEM OF OPTICAL ELEMENT SYNTHESIS. THE GEOMETRICAL OPTICS APPROXIMATION

The geometrical optics approximation may be obtained from the Fresnel approximation

$$u(x, y, f) = \frac{ke^{ikf}}{2\pi i}\int_G\int A(\xi, \eta)e^{ik[(x-\xi)^2/2f + (y-\eta)^2/2f + \varphi(\xi,\eta)]}\,d\xi\,d\eta$$

by taking $k \to \infty$.

The simplest case occurs when for each point (x, y) in the image region D there is only one stationary point (ξ_0, η_0), given by the equation of stationary phase:

$$\begin{cases} \varphi'_\xi + \left[\dfrac{(x-\xi)^2}{2f} + \dfrac{(y-\eta)^2}{2f}\right]'\xi = 0 \\ \varphi'_\eta + \left[\dfrac{(x-\xi)^2}{2f} + \dfrac{(y-\eta)^2}{2f}\right]'\eta = 0. \end{cases} \tag{9}$$

Equation (9) defines a mapping of the $0\xi\eta$ plane of region G to the plane $z = f$ (Fig. 2):

$$I_\varphi: \begin{cases} x = \xi + f\varphi'_\xi \\ y = \eta + f\varphi'_\eta \end{cases}. \tag{10}$$

Let $\|I_{\varphi'}\|$ denote the matrix $\begin{Vmatrix} x'_\xi & x'_\eta \\ y'_\xi & y'_\eta \end{Vmatrix}$. It can easily be verified that as $k \to \infty$, $u(x, y, f)$ assumes the asymptotic form

$$u(x, y, f) = A(\xi_0, \eta_0)\left|\frac{\partial(x, y)}{\partial(\xi, \eta)}\right|^{-1/2} + O\left(\frac{1}{\sqrt{k}}\right). \tag{11}$$

Here $\partial(x, y)/\partial(\xi, \eta)$ is the Jacobian of the transformation (10) evaluated at (ξ_0, η_0). The synthesis problem of (1) may now be reformulated as follows. We must find the one-to-one transformations

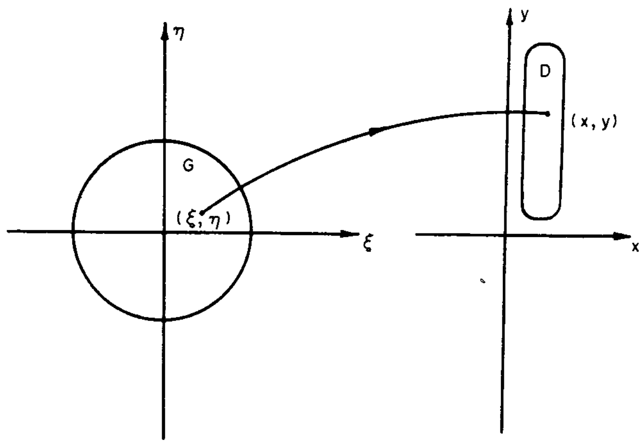

Fig. 2. To the formulation of problem 1 within geometrical optics.

$I: x = x(\xi, \eta)$, $y = y(\xi, \eta)$ which satisfy the condition $IG = D$, whereby

$$x'_\eta = y'_\xi;$$
$$|x'_\xi y'_\eta - x'_\eta y'_\xi| = g_0(\xi, \eta).$$

Here $g_0(\xi, \eta)$ is a function determined by the intensities of the incident and emergent images.

Suppose now that the solution of Eq. (9) is nonunique. This case corresponds to the presence of focused radiation. Consider the focusing problem (problem 3) in the focal curve $L: x = x_0(\theta)$, $y = y_0(\theta)$, $z = f$, where θ is a natural parameter. We assume that for each point $(x_0(\theta), y_0(\theta), f) \in L$, the manifold of stationary phase points (i.e. solutions of (9)) is a continuous curve $\Gamma(x_0(\theta), y_0(\theta)) = \Gamma(\theta)$, while for points $(x, y, f) \in L$, $\Gamma(\theta) = 0$. In that case one may prove the following:

Theorem 2 (necessary condition for focusing) [5]

Let $I_\varphi G = L$ and rang $\|I_{\varphi'}\| = 1$ everywhere in the region G (G is a strongly convex manifold). Then $\Gamma(\theta)$ is a curve segment with normal $\{\dot{x}_0(\theta), \dot{y}_0(\theta)\}$.

The preceding theorem considerably simplifies the inverse synthesis problem within the geometrical optics approximation.

For fields on the curve $(x, y, f) \in L$ one may obtain the $k \to \infty$ asymptotic form

$$u(x, y, f) = \sqrt{\frac{k}{2\pi}}\, e^{ik(f + S_0)}\, e^{i(\pi/4)} \int_{\Gamma(\theta)} \frac{A(\xi, \eta)\, dl}{\sqrt{|\mathrm{Sp}\, \|I_{\varphi'}\||}} + O(\sqrt{k}). \tag{11'}$$

Here $\mathrm{Sp}\,\|\cdot\|$ is the trace of the matrix $\|\cdot\|$. We can see that as $k \to \infty$ the field amplitude on the curve approaches infinity. For the field amplitude outside the curve (including ends) the following representation may be shown to hold:

$$u(x, y, f) = \frac{c(x, y)}{\sqrt{k}} + O\left(\frac{1}{\sqrt{k}}\right).$$

Suppose that at $x_0 = 0$, $y_0 = 0$ the curve $\mathscr{L}$ has the $0Y$ axis as its tangent. Consider the field $u(x, 0, f)$ along the $0X$ axis. In a real experiment one measures the intensity averaged over some interval:

$$I_{\mathrm{obs}} = \int_{-\delta}^{\delta} |u(x, 0, f)|^2\, dx.$$

We are led to

Theorem 3

As $k \to \infty$, the quantity I_{obs} has the asymptotic form

$$I_{\text{obs}} = \int_{\Gamma(\theta)} \frac{A^2(\xi, \eta)\,\mathrm{d}l}{|\mathrm{Sp}\,\|I_{\varphi}'(\xi, \eta)\|\,|} + O(1). \tag{12}$$

The result of this theorem may be reformulated as follows (*cf.* Fig. 1)

$$I_{\text{obs}} = \lim_{\delta' \to 0} \frac{1}{\delta'} \iint_{I_{\varphi}^{-1}(M(\theta)M(\theta+\delta'))} A^2(\xi, \eta)\,\mathrm{d}\xi\,\mathrm{d}\eta + O(1) = P_{I_\varphi}(\theta) + O(1). \tag{13}$$

Hence for large enough k, I_{obs} is simply a mean energy "rolled up" into an infinitesimally small arc $M(\theta)M(\theta + \delta')$. We shall use this definition of the focused radiation intensity in the following.

Problem 3. Focusing along a line

Let us discuss the mathematical formulation of the inverse problem for radiation focusing along a line $\mathscr{L}$. Let $p(\theta)$ be a piecewise continuous function, $0 < \theta < l_0$. For phase optical elements we are required to find a function $\varphi(\xi, \eta)$ such that the mapping I_φ,

$$I_\varphi : \begin{cases} x = \xi + f\varphi'_\xi \\ y = \eta + f\varphi'_\eta \end{cases} \qquad (\xi, \eta) \in G$$

maps region G into the line $\mathscr{L}$, i.e. $I_\varphi G = \mathscr{L}$.

In addition, $P_{I_\varphi}(\theta) = p(\theta)$, that is to say, the intensity distribution of the focused radiation $p(\theta)$ is given. The question is then whether the focusing problem into a line has a continuous solution. It turns out that a continuous solution does not always exist. Necessary and sufficient conditions can be formulated (on the curvature of the curve [5] and on the function $p(\theta)$) guaranteeing that continuous solutions exist. For unique continuous solutions not more than two are needed.

We can weaken the continuity conditions on the function φ in G. Let $\varphi \in C_2$ everywhere in G, with the exception of a smooth curve γ_{def}. Then the following theorem can be proved [6]:

Theorem 4

Let $\mathscr{L}$ be a piecewise continuous curve and $p(\theta)$ an arbitrary piecewise continuous function. Then the synthesis problem is solvable, i.e. there exists a function $\varphi \in C_2(G/\gamma_{\text{def}})$ such that $I_\varphi G = \mathscr{L}$ and $P_{I_\varphi} = p(\theta)$.

ALGORITHM FOR SOLVING THE LINE FOCUSING PROBLEM

Based on the foregoing results one can develop an algorithm for solving the problem of optical element synthesis when focusing radiation along a line. To start with, sufficient conditions must be given for the existence of continuous functions [5]. If such solutions do exist, the stationary family $\{\Gamma(\theta)\}$ is constructed [1]. If they do not, the curve γ_{def} is found and a stationary family is constructed in the region $G \backslash \gamma_{\text{def}}$ [6]. Once this stationary family has been found it signifies that the mapping I_φ in region G has been determined. Recovering the function φ from the relations

$$\begin{cases} x = \xi + f\varphi'_\xi \\ y = \eta + f\varphi'_\eta \end{cases} \qquad (\xi, \eta) \in G$$

is not difficult. In Figs 3 and 4 we illustrate the appropriate stationary sets, both when a continuous solution exists, and when it does not. Figures 5 and 6 show optical element masks when focusing Gaussian beams into a curve with uniformly distributed focused radiation.

Problem 4. Flat optics element design based on diffraction lattices (beam splitters, couplers, and so on)

Construction of optical systems such as beam splitters and couplers is one of the important problems of modern laser optics. Since the calculation of diffraction lattices may be reduced, to a good approximation, to the calculation of two-dimensional constructions, we may employ here a more complicated model based on Maxwell's equations.

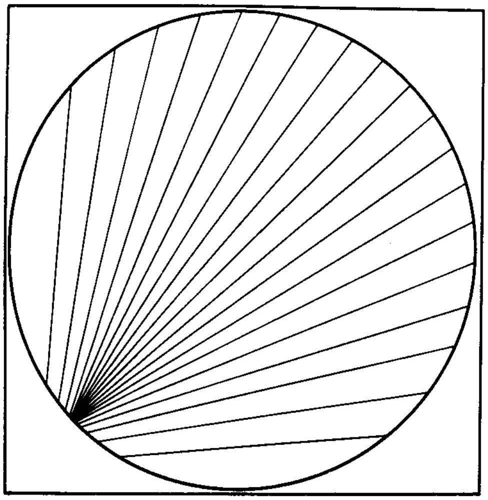

Fig. 3. Stationary set $\{\Gamma(\theta)\}$ when focusing problem has continuous solution.

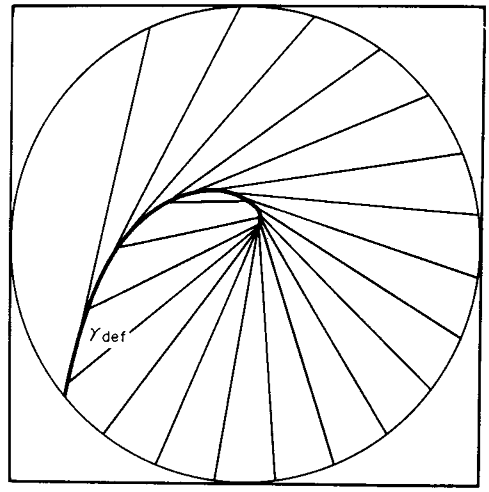

Fig. 4. Stationary set $\{\Gamma(\theta)\}$ in absence of continuous solution; the function $\varphi(\xi, \eta)$ has a discontinuity on the curve γ_{def}.

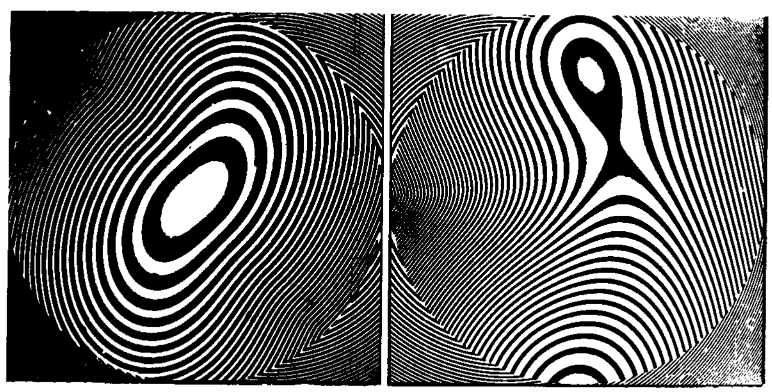

Fig. 5. Optical element masks (the optical density is proportional to the relief height of the phase optical element) for line focusing with uniformly distributed intensity of focused radiation. Continuous solution exists.

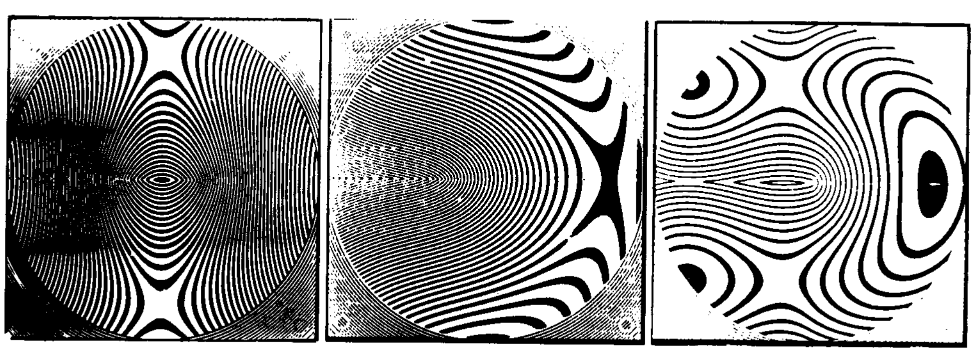

Fig. 6. Optical element masks for line focusing. No continuous solutions.

These approximations allow one to take into account effects of polarization, edges, finite conductivity, etc. A precise mathematical statement of the problem would take up too much space, and we shall therefore consider only some typical results, obtained in the Fresnel approximation via models based on the Maxwell approximation. The results indicate that if a sufficiently large number of wavelengths fit into a lattice period, then the various approximations give very similar results. If the lattice period d is comparable to the wavelength, then models based on the Fresnel

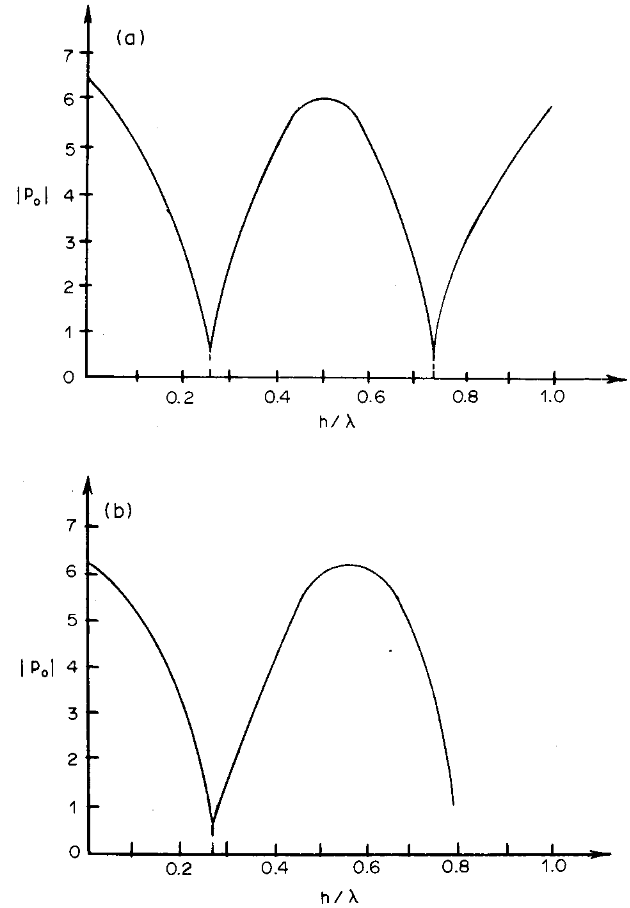

Fig. 7. Reflection amplitude of wave at $\theta = 0$ (normal to the lattice plane) as a function of h/λ: (a) $d/\lambda = 9.6$; (b) $d/\lambda = 2.6$.

approximation are no longer justified. To illustrate, we show in Fig. 7 graphs of the reflection amplitude p_0 with $\theta = 0$ (normal to the lattice) as a function of the lattice depth (λ is fixed).

We see that when $d/\lambda = 9.6$ the minimum reflection in zero order occurs at $h/\lambda = 0.25$. When $d/\lambda = 2.6$ the minimum is displaced (which does not occur in the Fresnel approximation) and the minimum reflection occurs at $h/\lambda = 0.28$.

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