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MODAL COMPOSITION OF RADIATION EXCITED BY LOCALIZED SOURCES WITH PARABOLIC REFRACTIVE INDEX PROFILE

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Abstract—Modal composition is studied for a waveguide with parabolic refractive index profile through the known directivity diagram of a localized source. Conditions have been determined under which the angular distribution of the localized source field is describable by a decomposition in terms of coherent states. Explicit expressions have been found for the coefficients of waveguide mode excitation, and for the case of non-directed sources a recursive relation was determined for their calculation. The mode composition of a dipole source is shown to be basically dependent on its position with respect to the waveguide axis. It is also demonstrated that, even in the case of a non-directed radiator, selective excitation of individual groups of waveguide modes is feasible.

The generation and control of wave fields in waveguides requires an understanding of the mode composition of directed and nondirected sources, as well as of the inherent antenna response in refracting waveguides. An important problem in this connection is the derivation of the initial energy distribution among the modes (actually given by the field amplitude distribution over the initial cross-section of the beam) which are excited by the sources with a known directivity diagram (DD) in a uniform medium.

This question has been discussed in a number of papers. For example, the mode excitation coefficients of normal waves in a uniform layer with boundaries have been calculated in [1, 2], while a WKB calculation for excitations of the underwater-sonic-channel (USC) in a shallow ocean has been performed, which took into account both the waveguide boundaries and the refraction of waves [3].

The aim of the present work is to examine the initial composition in a refracting waveguide of modes which are excited by localized sources of radiation. We depart from [1–3] in that the essence of our method is a field expansion in terms of exact solutions of a parabolic equation, in what amounts to a coherent state expansion [4, 5]. The problem is discussed in terms of a model of a multimode gradient waveguide with a quadratic refractive index distribution, which has been used in a number of important cases to describe natural waveguides [6, 7]. The approach is valid for deriving the low order ($m \ll M$) mode excitations which are responsible for energy transfer over very large distances, for example, in the USC of deep oceans or in the ionospheric waveguide channel (IWC) [6, 7]. It is especially convenient when the WRB approximation breaks down, in particular for the description of rays in the vicinity of the waveguide axis.

We consider a two-dimensional waveguide, uniform in the horizontal direction x, with a parabolic refractive index profile in the vertical z direction:

$$n^2(z) = n_0^2 - \omega^2 z^2, \tag{1}$$

where n_0 is the refractive index on the waveguide axis, and ω is the gradient parameter of the medium responsible for the transverse distribution of the refractive index.

A radiation source in such a guide may be considered localized, provided its linear dimension is much less than the width of the fundamental mode ($m = 0$), i.e.

$$l \ll \Delta(z_0) \tag{2}$$

$$\Delta(z_0) = (k\omega)^{-1/2} \tag{3}$$

where $k = 2\pi/\lambda$, λ being the wavelength of the radiation. Such sources can either be pointlike ($l \ll \lambda$) or extended ($l > \lambda$), and they may have a definite, albeit very narrow, DD.

In a uniform medium the radiation source field in the far zone (at distance $L > l^2/\lambda$) may be represented as a superposition of uniform plane waves [8], each of which is a solution to the scalar Helmholtz equation, and propagates along the trajectory of a geometrical ray. However, new features arise if the radiation sources are situated in a refracting waveguide. First of all, it has been shown [4], that in a medium with a parabolic index profile radiation propagation is conveniently described in terms of a coherent state [CS] expansion. The coherent states correspond to wave packets maximally localized in the phase space of the beam. The "centre of mass" of the wave packet propagates along geometrical ray paths, its widths defines the localization of the ray and is the same as the width of the fundamental mode (3) in the medium described by (1). Therefore, on physical grounds, it is natural to consider the DD as a function which describes the distribution of the source energy over the coherent states. From a mathematical point of view this is equivalent to replacing the original ray described by a uniform plane wave by a CS, i.e. by a nonuniform plane wave with a gaussian amplitude distribution, the latter representing a wave packet of uniform plane waves with some angular width.

Since a uniform plane wave is a mathematical abstraction, and in fact it possesses a certain width $(z)_{DD}$ (determined, for example, by the aperture of the receiver employed to measure the DD), the above replacement is completely legitimate if this transverse dimension is small compared to the width of the CS (i.e. the zeroth mode width), in other words, if

$$\Delta(z_0) \gg (z)_{DD} \tag{4}$$

or

$$\omega \ll (2k\Delta^2(z)_{DD})^{-1}. \tag{5}$$

If the DD is fixed within an angular error $\Delta(\omega)_{DD}$, then the right hand side of inequality (4) must be replaced by $\Delta(z)_{DD} = \lambda/(4\pi n \sin(\Delta(u)_{DD})$ as indicated by the uncertainty relation (4). The condition (4) is then equivalent to the requirement that the angular width $\Delta\omega$ of the CS does not exceed $\Delta\varphi_{DD}$: $\Delta\varphi \ll \Delta(\varphi)_{DD}$.

The second feature peculiar to the treatment of sources in such waveguides is that the very idea of a waveguide DD can only be applied if, for distances corresponding to the far zone, the transverse nonuniformities of the medium hardly affect the parameters of the radiation. The quantity which describes the characteristic distance over which the change in radiation parameters in the ray phase space is insignificant (i.e. stays less than its dispersion) is the stationarity length X_1^{α} introduced in [9]. For example, in a CS ray in medium (1) the stationarity length is the minimal distance along the axis over which the shift of the wave packet centre in the ray phase space does not exceed its width. Over distances of this order the medium may therefore be considered uniform, in the sense that nonuniformities do not affect the ray propagation significantly.

For medium (1) the stationarity length is given by the expression [9]:

$$X_1^{\alpha} = n_0/\omega|\alpha|, \tag{6}$$

where $\alpha = (k\omega/2)^{1/2} \cdot (z_0 + in \sin \varphi_0/\omega)$; z_0 is the initial $(x = 0)$ vertical coordinate of the ray, and φ_0 is the initial angle between the ray and the x-axis. The stationarity length decreases as α increases. Hence, if $\bar{\alpha}$ be the mean number of rays excited in the medium, such that $\bar{\alpha} \leqslant \alpha_{max}$, then the stationarity length of such a bundle is equal to $X_1^{\bar{\alpha}}$. The angular distribution of the field from the localized source is thus described by means of a DD which we denote by D_α (here and below the DD is understood to mean a source field expansion in terms of coherent states), provided

$$l^2/\lambda \ll X_1^{\bar{\alpha}}. \tag{7}$$

applies.

As a numerical example we assume that for sound propagation in the USC of a deep ocean with frequency $f = 100$ Hz and $\omega = 10^{-4}$ m^{-1}, the source may be considered localized when $l \ll 600$ m. A similar restriction holds for the aperture of the receiver. If the source is near the waveguide axis, then one may estimate the stationarity length from (6): $X_1^{\alpha} \simeq (2k\omega \sin^2 \varphi_{max})^{1/2}$. Since the maximum slope angle does not exceed $u = \pi/12$ for the USC in (6), we find $X_1^{\alpha} = 4.5 \times 10^3$ m. Accordingly, for localized sources satisfying (2), condition (7) applies, and the angular distribution of the energy of these sources is correctly described in terms of the DD.

Let us now analyse the composition of modes excited by sources possessing various DD. Let $D_\alpha(\theta; u)$ be the directivity diagram of a localized source situated at the point (x_0, z_0) of a waveguide whose effective width is h. The angle θ defines the direction of the centre of the DD with respect to the waveguide axis, while u is the angle between the ray and waveguide axis. It is known [4] that the redistribution of energy over the ray modes is given by the Poisson law:

$$|\langle m|\alpha\rangle^2 = |\alpha|^{2m} \exp(-|\alpha|^2)/m! \tag{8}$$

The mode excitation coefficients may then be represented in terms of an integral:

$$\varepsilon_m = \frac{\int (D_\alpha(\theta;\varphi)|\alpha|^{2m} \exp(-|\alpha|^2)/m!) d(\operatorname{Im}\alpha)}{\int D_\alpha(\theta;\varphi) d(\operatorname{Im}\alpha)}, \tag{9}$$

where

$$|\alpha|^2 = (k\omega)2)(z_0^2 + n^2 \sin^2 \varphi/\omega^2); \qquad d(\operatorname{Im}\alpha) = (k/2\omega) n \cos\varphi \, d\varphi .$$

Equation (9) characterizes the ratio of the energy stored in the given mode of the guide to the total energy radiated by the localized source.

As a concrete example consider a dipole source with $D_\alpha S(\Theta;u) = D_0 \cos^2(\Theta - u)$ within a nondirected radiation source with $D_\alpha(\Theta;u) = D_0$. Substituting D_α into (9), and performing the calculation, we obtain an explicit expression in terms of finite sums corresponding to the dipole and the nondirected source:

$$\varepsilon_m(\Theta;\xi_0) = (3\exp(-\xi_0^2)/\mu) \sum_{p=0}^{m} \frac{\Gamma(m-p+1/2)}{\Gamma(m-p+1)\Gamma(p+1)} \times \xi_0^{2p}[\Phi_1(\theta)A_{m-p} + (m-p+1/2)\mu^{-2}\Phi_2(\Theta)A_{m-p+1}] \tag{10a}$$

$$\varepsilon_m(\xi_0) = (\exp(-\xi_0^2)/\mu) \sum_{p=0}^{m} \frac{\Gamma(m-p+1/2)}{\Gamma(m-p+1)\Gamma(p+1)} \xi_0^{2p} A_{m-p}, \tag{10b}$$

where

$\xi_0 = \sqrt{\dfrac{k\omega}{2}}\, z_0$ is the dimensionless vertical coordinate of the source,

$\mu = (2kn_0^2/2\omega - \xi_0^2)^{1/2}$;

$\Phi_1(\Theta) = \cos^2(\Theta)/(\cos^2\Theta + 1)$;

$\Phi_2(\Theta) = \cos(2\Theta)/(\cos^2\Theta + 1)$;

$A_p = \operatorname{erf}(\mu) - \sum_{j=1}^{p} \dfrac{\mu^{2j-1}\exp(-\mu^2)}{\Gamma(j+1/2)}$;

$\operatorname{erf}(\mu) = 2\pi^{-1/2} \int_0^\mu \exp(-t^2)\,dt$ is the error function, and

$\Gamma(t)$ is the gamma function.

For sources located near the guide axis ($z_0 \ll n_0/\omega$) expressions (10a), (10b) simplify [10]:

$$\varepsilon_m(\Theta;\xi_0) = (3\exp(-\xi_0^2)/\mu_0) \sum_{p=0}^{m} \frac{\Gamma(m-p+3/2)}{\Gamma(m-p+1)\Gamma(p+1)} \xi_0^{2p}[\Phi_1(\Theta) + (m-p+1/2)\mu_0^{-2}\Phi_2(\Theta)] \tag{11a}$$

$$\varepsilon_m(\xi_0) = (\exp(-\xi_0^2)/\mu_0) \sum_{p=0}^{m} \frac{\Gamma(m-p+3/2)}{\Gamma(m-p+1)\Gamma(p+1)} \xi_0^{2p}, \tag{11b}$$

where $\mu_0 = (k/2\omega)^{1/2} n_0$.

The dependence of the mode excitation coefficients for a dipole source on the direction of the DD is characterized by the first term in square brackets of (11a) which does not depend on the mode number. The second term gives contributions which are important for $\Theta \sim \pi/2$, and in general depends on the mode number as well as the rest of the parameters (x_0, λ, ω). These contributions have been studied in more detail in [10]. We only note that when $\Theta = \pi/2$ the mode excitation coefficients are small—this is the weakly-excited regime of the waveguide. On the other hand, a change in direction of the DD (the term $\Phi_2(\Theta)$ may be neglected) does not lead to a redistribution

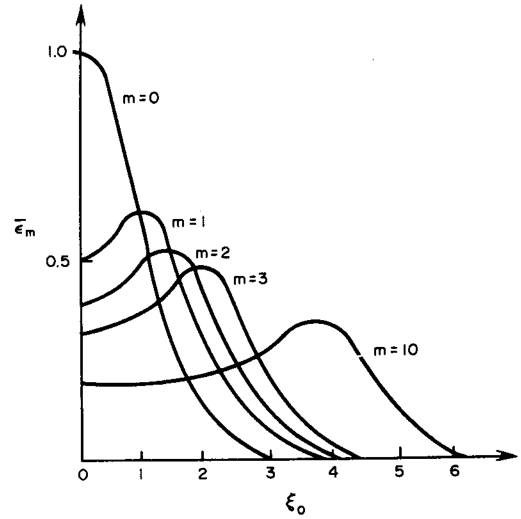

Fig. 1.

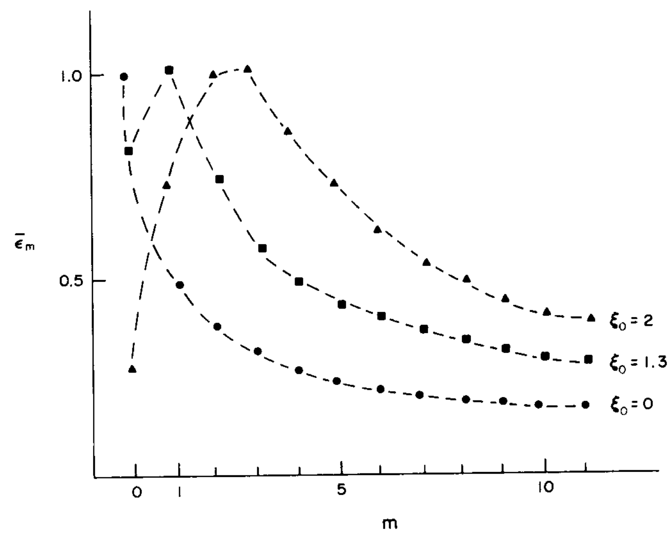

Fig. 2.

of energy among the modes, but only to a general diminution of the energy of the low-lying modes trapped by the waveguide. In addition, if the DD direction is fixed, Eqs (11a) and (11b) are equivalent, so that we may further restrict discussion to a nondirected source.

Figure 1 shows the dependence of the excitation coefficients, normalized to $\varepsilon_0(0)$, on the source distance along the waveguide axis. We notice a characteristic shift in the maxima of the excited modes along the guide axis as the mode number increases. It can be seen therefore, that even for nondirected radiators modes are excited nonuniformly. For example, for an axial source the excitation coefficients decrease with increasing mode number.

Therefore, one may in principle selectively excite separate groups of modes of nondirected radiators. As an illustration of this possibility, Fig. 2 shows the dependence of the coefficients of excitation on the mode number for various source positions along the waveguide axis. It is clear from the curves that the nonuniformity of mode excitation weakens as the source is displaced along the axis.

In conclusion we give a recurrence relation which simplifies the calculation of the mode excitation coefficient for a nondirected source:

$$\varepsilon_m(\xi_0)\left(\sum_{p=0}^{m-1} \varepsilon_p(\xi_0) + \xi_0^2\varepsilon_{m-1}(\xi_0)\right)\Big/2m, \quad m = 1, 2, \ldots \tag{12}$$

It reduces to $\varepsilon_m(0) = [(2m-1)/2m]\varepsilon_{m-1}(0)$ for an axial source.

We have seen that the method presented enables one to calculate explicitly the excitation coefficients which determine the initial mode composition. The method may also be a useful alternative to the WKB approximation for investigating excitations in natural waveguides.

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